MAT116 Final Review Session Chapter 3: Polynomial and Rational Functions

Quadratic Function

- A **quadratic function** is defined by a quadratic or second-degree polynomial.

  - Standard Form
    - $f(x) = ax^2 + bx + c$, where $a \neq 0$.

  - Vertex Form
    - $f(x) = a(x-h)^2 + k$, *where* $a \neq 0$.

Vertex and Axis of Symmetry

- The point (h, k) is the vertex of the parabola if it is in vertex form.

- The point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ is the vertex of the parabola if it is in standard form.

- The vertical line $\mathrm{x} = -\frac{b}{2a}$ is called the **axis of symmetry** for the graph of $f(x) = ax^2 + bx + c$.

Examples: Find the Following

1. $f(x) = 2(x-3)^2 + 5$

   Vertex:

2. $g(x) = 3x^2 - 5x + 2$

   Vertex:

   Axis of Symmetry:

Opening and Maximum and Minimum

- If ***a > 0***, the graph of *a quadratic function* **opens upward**
- If ***a* < 0**, the graph **opens downward**
- If ***a > 0***, k is the **minimum** value of the function
- If ***a < 0***, k is the **maximum** value of the function

Examples: Find the min or max and state if it opens up or down.

3. $h(x) = -2x^2 + 4x + 7$

Intercepts

- The **y-intercept** is found by letting x = 0 and solving for y.
  - Write as an ordered pair: (0 , y)

- The **x-intercepts** are found by letting y = 0 and solving for x.
  - Solve by factoring, square roots, or the quadratic formula.
  - Write as an ordered pair: (x , 0)

Examples: Find the x and y intercepts.

4. $h(x) = -2x^2 + 4x + 7$

5. $f(x) = 2(x-3)^2 + 5$

Examples: Identify the following.

6. $y = 2x^2 + 8x + 6$

- Vertex:
- Axis of Symmetry:
- Minimum:
- Maximum:
- Y-intercept:
- X-intercepts:
- Vertex form:

Quadratic Inequalities

Strategy: Solving a Quadratic Inequality by the Graphical Method

1. Get 0 on one side of the inequality and a quadratic polynomial on the other side.
2. Find all roots to the quadratic polynomial.
3. Graph the corresponding quadratic function. The roots found in step (2) determine the x-intercepts.
4. Read the solution set to the inequality from the graph of the parabola.

Examples: Solve the inequality and graph it. Write your answer in interval notation.

7. $x^2 + 6x > -8$

8. $2x + 15 < x^2$

Zeroes of Polynomial Functions

- Division of Polynomials
  - **Long Division** – can be used to divide any two polynomials
  - **Synthetic Division** - Can only be used to divide two polynomials when dividing by x-k.

- **Remainder Theorem**
  - If R is the remainder when a polynomial $P(x)$ is divided by $x – c$, then $R = P(c)$.

Examples: Solve.

9. $(x^3 + 5x^2 - 3x + 15) \div (x^2 - 2)$

10. $\frac{x^2+5x+6}{x+2}$

11. If $h(x) = -3x^3 + 5x^2 - 6x + 1$, use the Remainder Theorem to find $h(-1)$.

Rational Zero Theorem

If

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial function with integral coefficients ($a_n \neq 0$ and $a_0 \neq 0$) and $\frac{p}{q}$ (in lowest terms) is a rational zero of $f(x)$, then p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

- To find the rational zeros, divide all the factors of the constant term by all the factors of the lead coefficient.

Examples: List all possible rational roots and find all the real and imaginary zeroes.

12. $h(x) = x^3 - x^2 - 7x + 15$

Theory of Equations

- **Multiplicity:** If the factor $x - c$ occurs k times in the complete factorization of the polynomial $P(x)$, then c is called a root of $P(x) = 0$ with **multiplicity *k***.
  - Multiplicity is the number of times a zero occurs.

- **Conjugate Pairs Theorem:** If $P(x) = 0$ is a polynomial equation with real coefficients and the complex number $a + bi$ ($b \neq 0$) is a root, then $a - bi$ is also a root

Examples: State the degree, find all real and imaginary roots and state their multiplicities.

13. $f(x) = x^5 - 6x^4 + 9x^3$

Examples: Find a polynomial with the given roots.

14. *12, 8i*

15. *5, 4-3i*

Symmetry

- Symmetric about the y-axis: f(x) is an **even function** if f(-x) = f(x)
- Symmetric about the origin: f(x) is an **odd function** if f(-x) = -f(x)
- A quadratic function is symmetric about the axis of symmetry if x = $-\frac{b}{2a}$

Examples: State whether the function is even, odd or neither.

16. $f(x) = x^6 - x^4 + x^2 - 8$

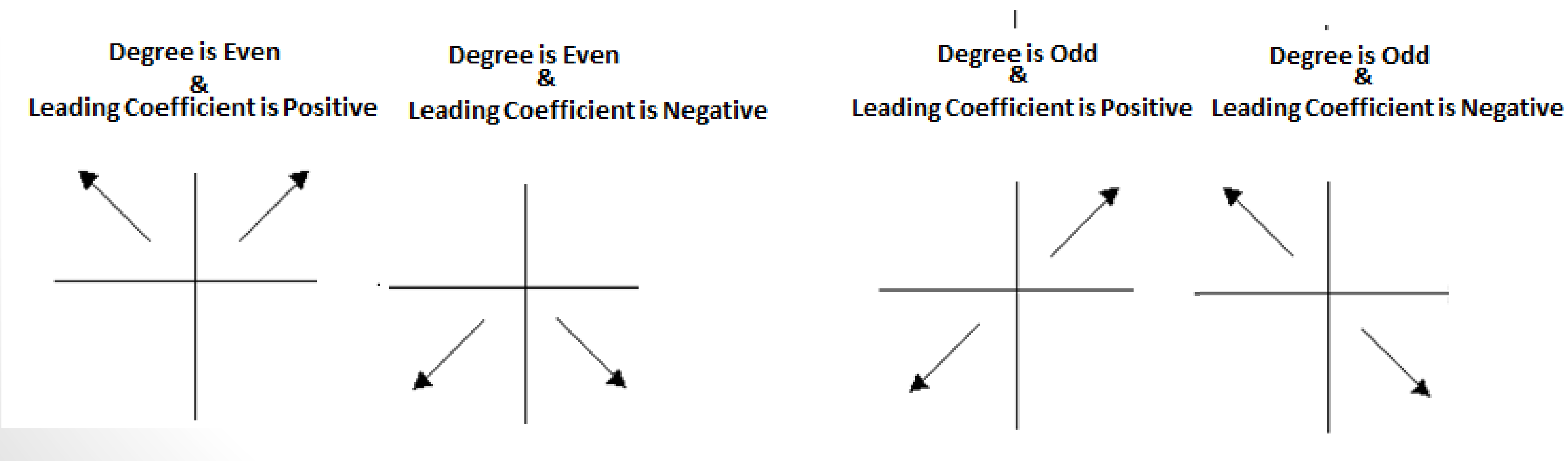
17. $g(x) = 4x^3 - x$

18. $h(x) = x^3 - x^2 + 2$

Behavior

- Multiplicity of Zeroes
  - **Even Multiplicity**: the graph touches but does NOT cross the x-axis at the x-intercept
  - **Odd Multiplicity**: the graph crosses the x-axis at the x-intercept

- The **Leading Coefficient Test** helps to determine the end behavior of a graph.



Examples: Graph the following.

19. $f(x) = x^3 - 3x^2$

20. $h(x) = x^6 + 2x^5 + x^4$

Polynomial Inequalities

- Very similar to solving Quadratic Inequalities.

Strategy: Solving a Polynomial Inequality by the Graphical Method

1. Get 0 on one side of the inequality and a polynomial on the other side.
2. Find all roots to the polynomial.
3. Graph the corresponding function. The roots found in step (2) determine the *x*-intercepts.

Examples: Solve and write your answer in interval notation.

21. $x^3 + 4x^2 - x - 4 > 0$

22. $x^3 + 2x^2 - 2x - 4 < 0$

Rational Functions

- If P(x) and Q(x) are polynomials, then a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

  is called a rational function, provided that Q(x) is not the zero polynomial.

Asymptotes

- An **asymptote** is an "invisible" line that the function is always approaching but never reaching.
- **Vertical asymptotes** correspond to where Q(x)= 0.
- **Horizontal Asymptotes**
  - If the numerator has a lower degree than the denominator, the horizontal asymptote is the line y=0.
  - If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is the line y=a/b where a is the lead coefficient of the numerator and b is the lead coefficient of the denominator.
- **Oblique (Slant) Asymptote**
  - If the degree of the numerator is one degree higher than the degree of the denominator, the graph of the function has an oblique asymptote. Divide the numerator by the denominator, the quotient (without the remainder) is your oblique asymptote.

Examples: Find the asymptotes.

23. $f(x) = \frac{-2x}{x^2+6x+9}$

24. $f(x) = \frac{2x^2-11}{x^2-9}$

25. $f(x) = \frac{x^2-6x+7}{x+5}$

Examples: Solve the Rational Inequality.

26. $\frac{x-1}{x-3} > 0$

27. $\frac{x+3}{x-2} < 0$

Miscellaneous Equations

- **Equations involving <u>absolute value</u>** can include more than one absolute value or contain higher degree polynomials where the definition for absolute value is used to determine the solutions.

- **Equations involving <u>square roots</u>** are solved by *squaring both sides* once a radical is isolated on one side of the equation.

- **Equations with <u>rational exponents</u>** are solved by raising both sides of the equation to a *reciprocal power* and considering positive and negative possibilities for even roots.

- **Equations of <u>quadratic type</u>** can be solved by *substituting a single variable* for a more complicated expression.

- **<u>Factoring</u>** is often the fastest method for solving an equation.

Examples: Solve.

Absolute Value Examples:

28. $|v^2 - 3v| = 5v$

29. $|x + 5| = |2x + 1|$

30. $|x - 4| - 1 = -4x$

Examples: Solve.

Square Root Examples:

31. $\sqrt{x+1} = x - 5$

32. $\frac{1}{z} = \frac{3}{\sqrt{4z+1}}$

33. $\sqrt{x+40} - \sqrt{x} = 4$

Examples: Solve

Rational Exponent Examples:

34. $x^{\frac{2}{3}} = 2$

35. $w^{-\frac{3}{2}} = 27$

36. $(t-1)^{-\frac{1}{2}} = \frac{1}{2}$

Examples: Solve.

Quadratic Type Examples:

37. $x^4 + 6x^2 - 7 = 0$

38. $x^4 - x^2 - 12 = 0$

39. $x - 7x^{\frac{1}{2}} + 12 = 0$

Examples: Solve.

Solving Higher Degree Polynomials with Factoring:

40. $2 + x - 2x^2 = x^3$

41. $2x^3 + 1000x^2 - x - 500 = 0$

42. $x^4 - 81 = 0$

Chapter 3 Review

- Quadratic Function
- Theory of Equations
- Rational Functions
- Miscellaneous Equations

Example Solutions

1) vertex = (3,5)
2) Vertex = $\left(\frac{5}{6}, -\frac{1}{12}\right)$ AoS = $x = \frac{5}{6}$
3) Max = 9, opens down
4) Y-int. = $(0,7)$, x-int. = $\left(\frac{2\pm3\sqrt{2}}{2}, 0\right)$
5) Y-int. = (0,23), x-int.= $\left(\frac{6\pm i\sqrt{10}}{2}, 0\right)$
6) Vertex = $(-2,-2)$, AoS = $x = -2$, min = -2, $y-int. = (0,16)$, x-int. = $(-3,0), (-1,0)$, Vertex Form = $2(x+2)^2 - 2$
7) $(-\infty, -4)U(-2, \infty)$
8) $(-\infty, -3)U(5, \infty)$
9) $x + 5\ R{:}\ -x + 25$
10) $x + 3\ R{:}\ 0$
11) $h(-1) = 15$
12) $\frac{p}{q} = \pm\frac{1,3,5,15}{1}, x = \{-3, 2, \pm i\}$
13) degree: 5, x = 0 multiplicity 3, x = 3 multiplicity 2
14) $x^3 - 12x^2 + 64x - 768$
15) $x^3 - 13x^2 + 65x - 125$
16) Even
17) Odd
18) Neither
19) Graph
20) Graph
21) $(-4, -1)U(1, \infty)$
22) $(-\infty, -2)U\left(-\sqrt{2}, \sqrt{2}\right)$
23) $x = -3, y = 0$
24) $x = 3, x = -3, y = 2$
25) $x = -5, y = x - 11$
26) $(-\infty, 1)U(3, \infty)$
27) (-3,2)
28) v = 0,8,-2
29) x = -2, 4
30) x = -1, 1
31) x = 8,3
32) z = $\frac{2\pm\sqrt{13}}{9}$
33) x = 9
34) x = $2\sqrt{2}$
35) w = 1/9
36) t = 5
37) $x = \pm1, \pm i\sqrt{7}$
38) x = $\pm2, \pm i\sqrt{3}$
39) x = 16, 9
40) x = 1, -1, -2
41) x = $-500, \pm\frac{1}{\sqrt{2}}$
42) x = -3, 3