MAT116 Final Review Session Chapter 4: Exponential and Logarithmic Equations, Section 5.1,5.5: Systems of Linear Equations and Inequalities, and Section 6.1-6.3: Matrices

Exponential Functions

Definition: Exponential Function

An **exponential function** with **base *a*** is a function of the form

$$f(x) = a^x,$$

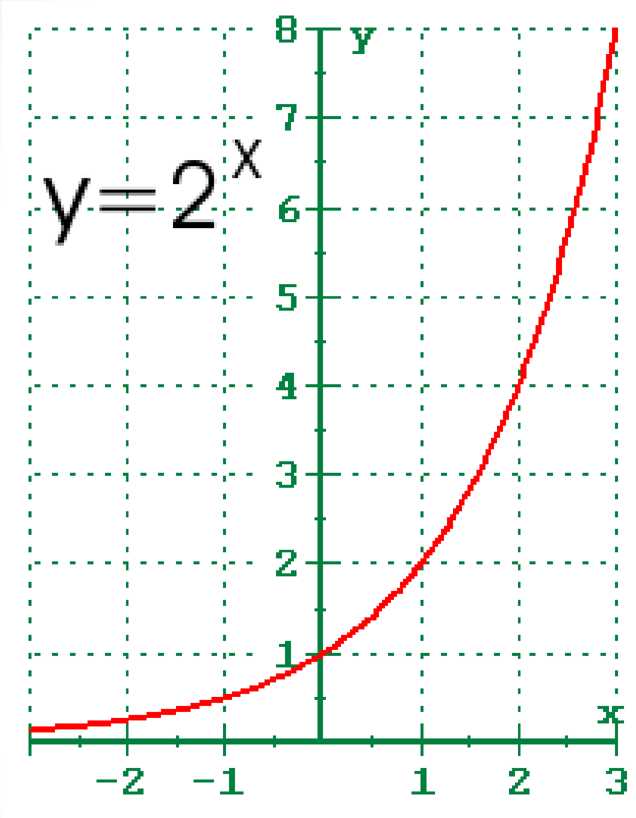
where a and x are real numbers such that $a > 0$ and $a \neq 1$.

Domain of an Exponential Function

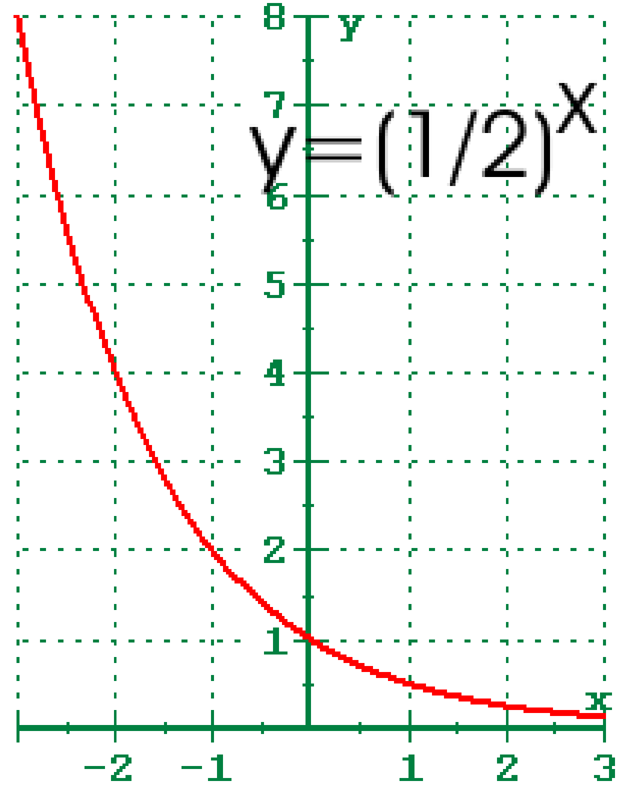
The **domain** of $f(x) = a^x$ for $a > 0$ and $a \neq 1$ is the set of all real numbers.

The Exponential Family of Functions

- Any function of the form $g(x) = b \cdot a^{x-h} + k$ is a member of the **exponential family** of functions.
  - The graph of *f* moves to the left if $h < 0$ or to the right if $h > 0$.
  - The graph of *f* moves upward if $k > 0$ or downward if $k < 0$.
  - The graph of *f* is stretched if $b > 1$ and shrunk if $0 < b < 1$.
  - The graph of *f* is reflected in the *x*-axis if *b* is negative.



Increasing function when a > 1.



Decreasing function when 0 < a < 1.

Logarithmic Functions

- Since exponential functions are one-to-one functions, they are invertible. The inverses of the exponential functions are called **logarithmic functions**.

- If $f(x) = a^x$, then instead of $f^{-1}(x)$, we write $log_a(x)$ for the inverse of the base-*a* exponential function.

- We read $log_a(x)$ as "log base *a of x*," and we call the expression $log_a(x)$ a **logarithm**.

Logarithmic Functions

Definition: Logarithmic Function

For $a > 0$ and $a \neq 1$, the **logarithmic function with base** ***a*** is denoted as $f(x) = log_a(x)$, where

$$y = log_a(x) \quad \text{if and only if} \quad a^y = x.$$

<u>Logarithmic Form</u>

$$y = \log_a x$$

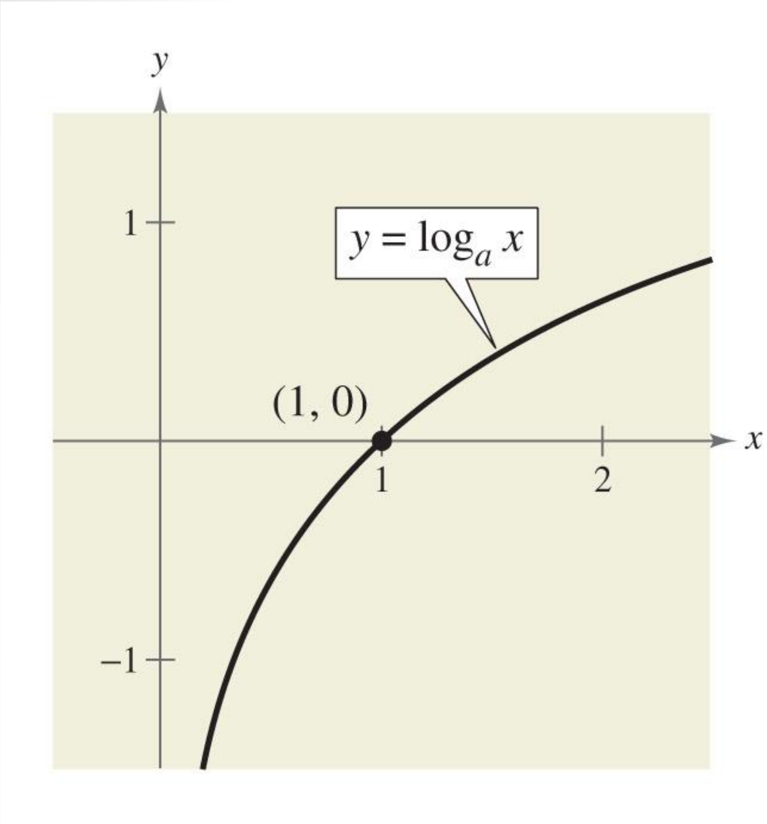
<u>Exponential Form</u>

$$a^y = x$$

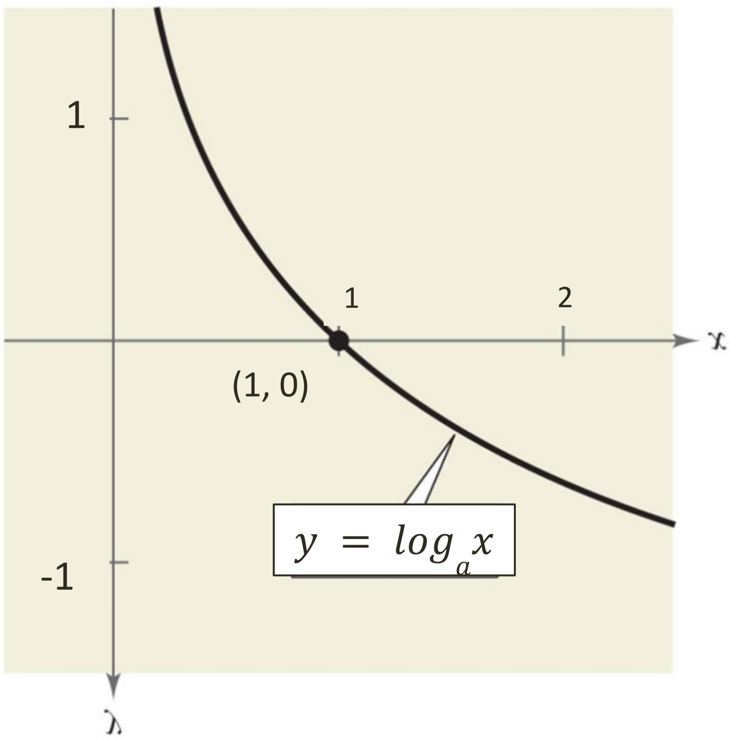
<u>**EQUIVALENT!**</u>

The Logarithmic Family of Functions

- Any function of the form $g(x) = b \cdot log_a(x - h) + k$ is a member of the **logarithmic family** of functions.
- The graph of *f* moves to the left if $h < 0$ or to the right if $h > 0$.
- The graph of *f* moves upward if $k > 0$ or downward if $k < 0$.
- The graph of *f* is stretched if $b > 1$ and shrunk if $0 < b < 1$.
- The graph of *f* is reflected in the *x*-axis if *b* is negative.



For $a > 1$



For $0 < a < 1$

Rules of Logarithms

- Change of base:

- $log_a x = \frac{log x}{log a} = \frac{ln x}{ln a} = \frac{log_b x}{log_b a}$

Examples: Solve. Round your answer to the nearest thousandths.

1. $log_3 16$

2. $log_{50} 5$

Properties of Logarithms

Product Property	$\log_a(uv) = \log_a u + \log_a v$
Quotient Property	$\log_a\left(\dfrac{u}{v}\right) = \log_a u - \log_a v$
Power Property	$\log_a u^n = n\log_a u$

Examples: Condense or Expand.

3. $log \frac{\sqrt{x}y^4}{z^4}$

4. $ln\sqrt[3]{5}x^2$

5. $2\ ln8 + 5\ ln(z-4)$

6. $2[3logx - log6 - logy]$

One-to-one Property

One-to-One Property of Exponential Functions

For $a > 0$ and $a \neq 1$,

if $a^{x_1} = a^{x_2}$, then $x_1 = x_2$.

One-to-One Property of Logarithms

For $a > 0$ and $a \neq 1$,

if $log_a(x_1) = log_a(x_2)$, then $x_1 = x_2$.

Examples: Find the exact value without a calculator.

7. $log_7 \sqrt[5]{7}$

8. $ln\ e^{12} + ln\ e^5$

9. $log_3 81^{-0.2}$

Examples: Find the exact value without a calculator.

10. $4(3^x) = 64$

11. $e^{-x^2} = e^{5x+6}$

12. $5 - 3e^x = 2$

13. $6(2^{x+5}) + 4 = 11$

14. $lnx = \frac{2}{3}$

15. $log_4(3x + 2) = log_4(6 - x)$

Examples: Solve for x.

16. $log_3(5x + 13) - log_3 6 = log_3 3x$

17. $logx + \log(x - 9) = 1$

18. $log_4(x) - log_4(x + 2) = 2$

19. $log(x + 1) - logx = 3$

20. $log4 = 1 + \log(x - 1)$

21. $6^x = 3^{x+1}$

22. $lnx + ln(x + 2) = ln8$

Strategies for Solving Exponential and Logarithmic Equations

- Try to use the one-to-one properties (match them up)
- Condense to single logarithms when possible
- Switch from exponential form to logarithmic form
- Switch from logarithmic form to exponential form

Solving a System of Equations

- Any collection of two or more equations is called a **system of equations.**

- The **solution set** of a system of two linear equations in two variables is the set of all ordered pairs that satisfy *both* equations of the system.
  - The graph of an equation shows all ordered pairs that satisfy the equation, so we can solve some systems by graphing the equations and observing which points (if any) satisfy all of the equations.

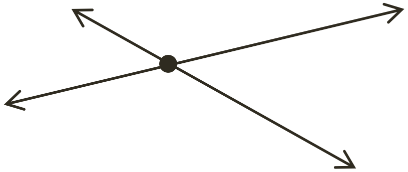
THESE SOLUTION WOULD BE WHERE THE GRAPHS INTERSECT.

Types of Systems

- A system of equations that has at least one solution is **consistent**. Two types of consistent systems are:

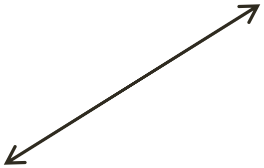
  - **Independent -** with <u>exactly one solution</u>

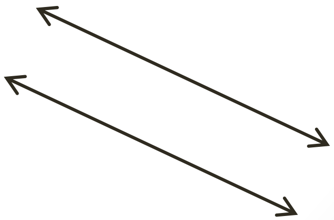
    Two Intersecting Lines



  - **Dependent -** with <u>infinitely many solutions</u>

    Same Lines





- A system with <u>no solutions</u> is **inconsistent**.

  Parallel Lines

The Substitution Method and the Addition Method

- In the **substitution method**, we eliminate a variable from one equation by substituting an expression for that variable from the other equation.

- In the **addition method**, (also called the elimination method) we eliminate a variable by adding the two equations.

    - It might be necessary to multiply each equation by an appropriate number so that a variable will be eliminated by this addition.

Examples:
Solve each System of Equations Using Substitution

25. $y = 2x + 1$
$3x - 4y = 1$

26. $y - 3x = 5$
$3(x - 1) = y - 2$

27. $2x + y = 9$
$4x + 2y = 18$

Examples:
Solve each System of Equations Using Addition/Elimination

28. $$y - 2x = 1$$
$$-4y + 3x = 1$$

29. $$y - 3x = 5$$
$$3(x - 1) = y - 2$$

30. $$2x + y = 9$$
$$4x + 2y = 18$$

Applications of Systems

- 31. Amy has a higher income than Vince, and their total income is \$82,000. If their salaries differ by \$16,000, then what is the income of each?

- 32. The Springfield zoo has different admission prices for adults and children. When Mr. and Mrs. Weaver went with their five children, then bill was \$33. If Mrs. Wong and her three children got in for \$18.50, then what were the individual prices for adult and children's tickets?

Inequalities and Systems of Inequalities

- If A, B, and C are real numbers with A and B not both zero, then

$$Ax + By < C$$

is called a linear inequality in two variables. In place of $<$ we can also use the symbols $\leq, >, \text{or } \geq$.

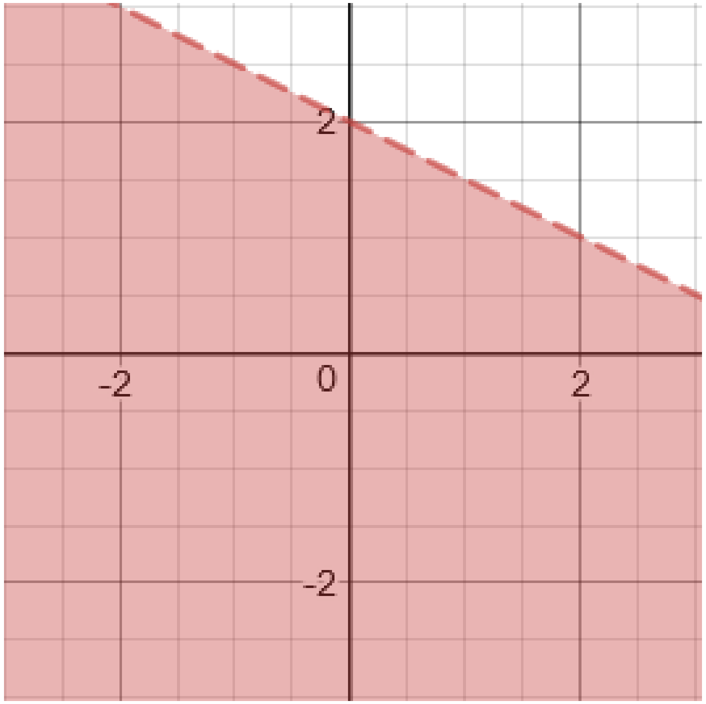
- An ordered pair (a, b) is a solution to an inequality if the inequality is true when x is replaced by a and y is replaced by b.
- The solution set for an inequality is the set of all points that make the inequality a true statement. We can represent the set with a graph.

Graphing Linear Inequalities

- First, solve the inequality for y, and graph the line.
  - If the inequality is $\leq or \ \geq$ use a solid line
  - If the inequality is $< or \ >$ use a dashed line
- Pick a test point and plug it into the inequality to see if it holds true.
  - If the test point gives a true statement, then shade the area the test point is in.
  - If the test point gives a false statement, then shade the area on the other side of the line.

Examples: Graph the Linear Inequalities

- $y < -\frac{1}{2}x + 2$



Solving a System of Linear Inequalities

- The solution set to a system of linear inequalities in two variables consists of all ordered pairs that satisfy ALL of the inequalities in the system.

- To find the solution set to a system, we graph the equation corresponding to each inequality in the system and then test a point in each region to see whether it satisfies all inequalities of the system.

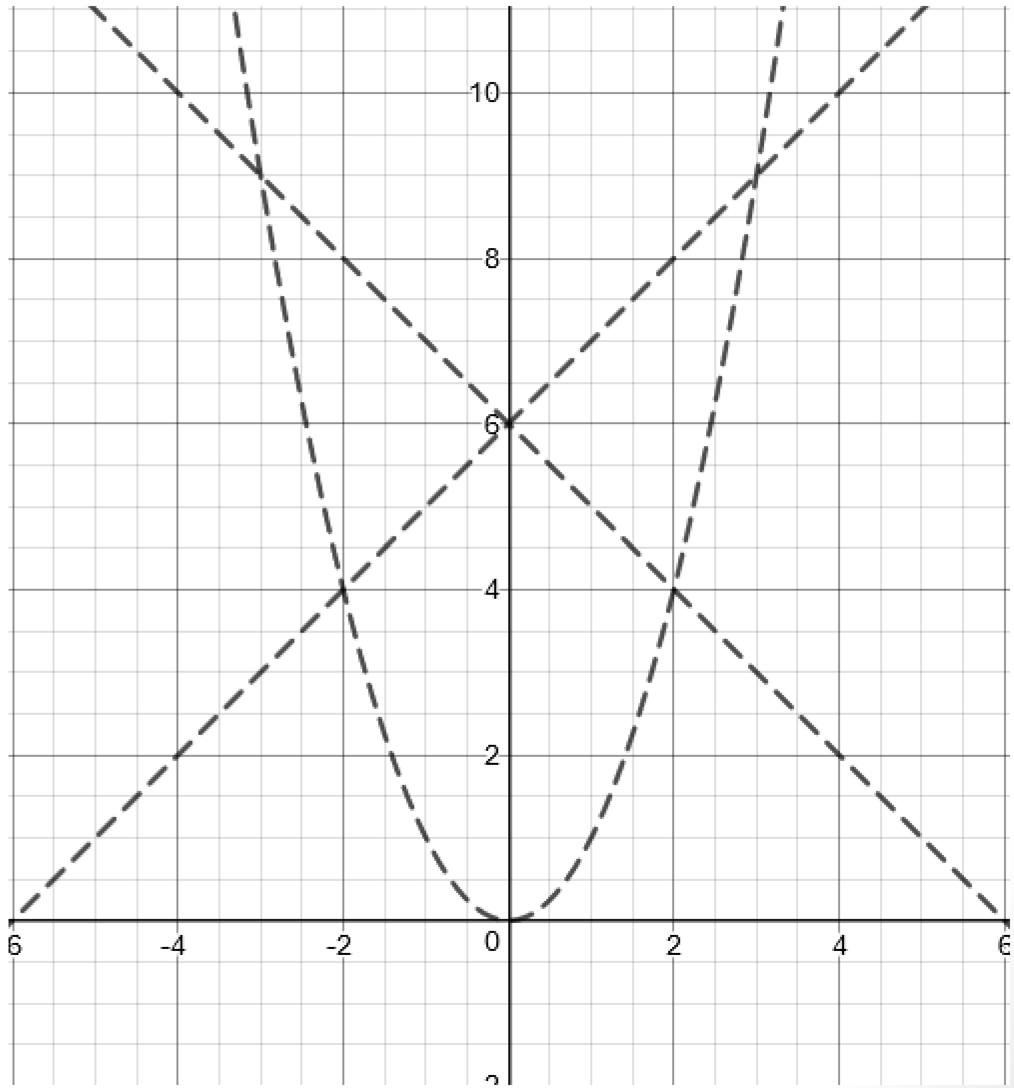
Example:

- $y > x^2$
- $y < x + 6$
- $y < -x + 6$

First graph each equation

$\rightarrow$

Then pick test points for each area



Example:

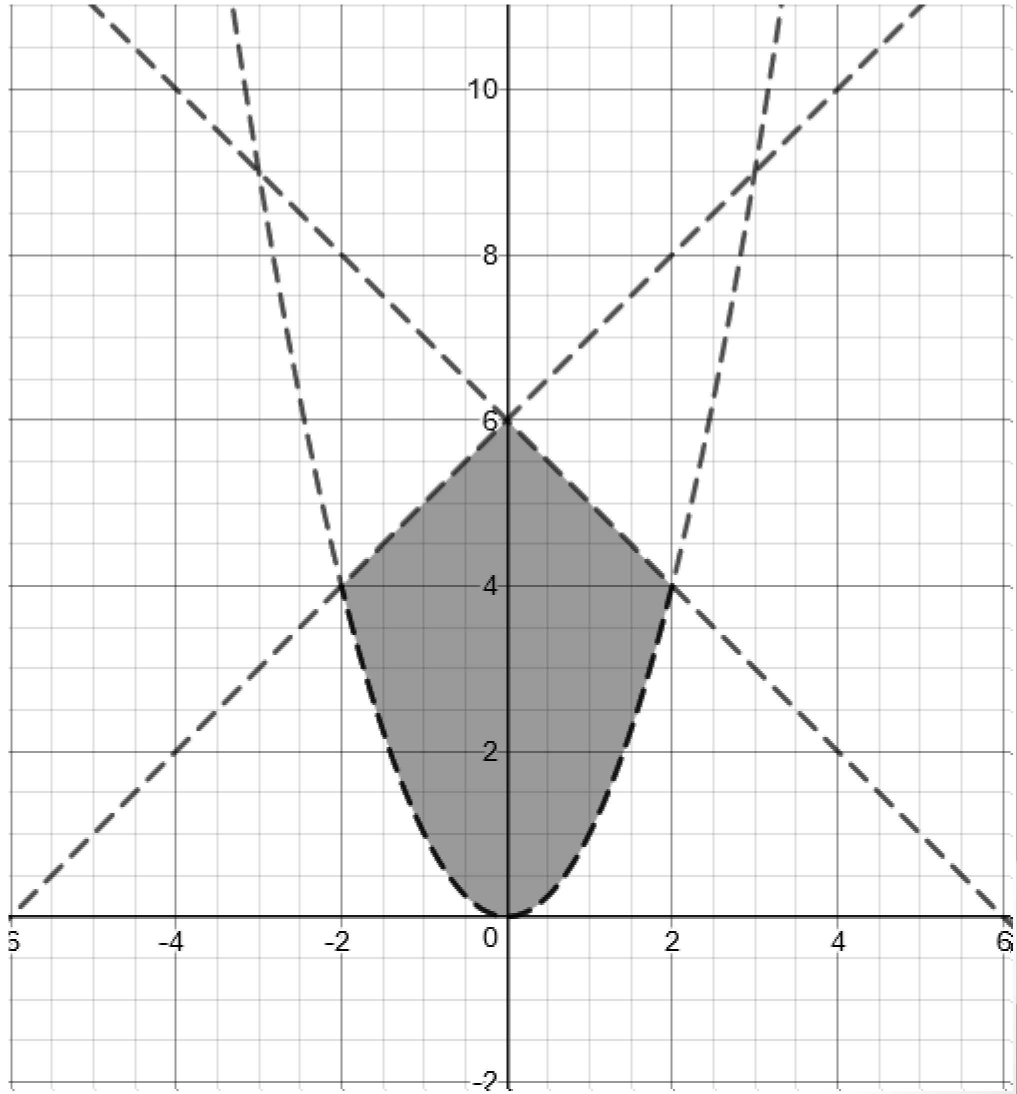
- $y > x^2$
- $y < x + 6$
- $y < -x + 6$

First graph each equation

$\rightarrow$

Then pick test points for each area

The test point that gives a true statement will be in the shaded area.



Solving Systems with Matrices in Your Graphing Calculator

- A **matrix** is a rectangular array of real numbers.
- The **rows** of a matrix run horizontally, and the **columns** run vertically.
- A matrix with *m* rows and *n* columns has **size** $m \times n$ (read "*m* by *n*").
  - The number of rows is always given first.

$$\begin{array}{cc} & \text{Columns} \\ \text{Rows} & \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \end{array}$$

This is a 2 x 3 matrix.

Converting a System of Equations into an Augmented Matrix

Both equations in the system need to be in standard form:
$Ax + By = C$

$$2x - 3y = 6 \xrightarrow{\text{Already in standard form.}} 2x - 3y = 6$$

$$y = x + 4 \xrightarrow[\text{Put into standard form.}]{} x - y = -4$$

$$\begin{aligned} 2x - 3y &= 6 \\ x - y &= -4 \end{aligned} \longrightarrow \begin{bmatrix} 2 & -3 & 6 \\ 1 & -1 & -4 \end{bmatrix}$$

Put the values of A, B and C into the matrix.

Row Operations

Any of the following row operations on an augmented matrix gives an equivalent augmented matrix.

- Interchanging two rows of the matrix
- Multiplying every entry in a row by the same nonzero real number
- Adding to a row a nonzero multiple of another row.

Examples: Solve by matrices.

33. $$-2\mathrm{x} + y = 1$$
$$3x - 4y = 1$$

34. $$y - 3x = 5$$
$$3x - 3 = y - 2$$

35. $$2x + y = 9$$
$$4x + 2y = 18$$

Operations with Matrices

- If two matrices are equal, then the corresponding entries are equal.
- The sum of two $m \times n$ matrices A and B is the $m \times n$ matrix denoted $A + B$ whose entries are the sums of the corresponding entries of A and B.
$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$$
- The additive inverse of A, denoted by $-A$ is found by changing the sign of every entry in matrix A.
*Note $A + (-A)$ is the zero matrix.
- The difference of two $m \times n$ matrices A and B is the $m \times n$ matrix denoted $A - B$, where $A - B = A + (-B)$.
- If A is an $m \times n$ matrix and b is a scaler, then the matrix bA is the $m \times n$ matrix obtained by multiplying each entry of A by the real number b.

Examples

36. $\begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ -2 & 3 \end{bmatrix}$

37. $\begin{bmatrix} -3 & 5 \\ 6 & 7 \\ 15 & -12 \end{bmatrix} - \begin{bmatrix} 1 & -7 \\ -5 & -4 \\ 18 & 8 \end{bmatrix}$

38. $2\begin{bmatrix} -1 & 7 \\ 9 & 6 \end{bmatrix}$

Multiplication of Matrices

- The product of an $m \times n$ matrix A and an $\mathrm{n} \times p$ matrix B is an $m \times p$ matrix AB whose entries are found as follows. The entry in the *i*th row and the *j*th column od AB is found by multiplying each entry in the *i*th of A by the corresponding entry in the *j*th column of B and adding the results.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \cdot \begin{bmatrix} b_1 & b_4 \\ b_2 & b_5 \\ b_3 & b_6 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 & a_1 \cdot b_4 + a_2 \cdot b_5 + a_3 \cdot b_6 \\ a_4 \cdot b_1 + a_5 \cdot b_2 + a_6 \cdot b_3 & a_4 \cdot b_4 + a_5 \cdot b_5 + a_6 \cdot b_6 \end{bmatrix}$$

$$2 \times 3\, matrix \cdot 3 \times 2\, matrix = 2 \times 2\, matrix$$

Examples

39. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$

41. $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$

Chapter 4, 5.1, 6.1 Review

- Exponentials and Logarithms
  - Functions
  - Solving
- Properties of Logarithms
- Applications
- Solving Systems of Equations

Example Answers

- 1) $\frac{log16}{log3}$
- 2) $\frac{log5}{log50}$
- 3) $\frac{1}{2}\log(x) + 4\log(y) - 4\log(z)$
- 4) $\frac{1}{3}\ln(5) + 2\ln(x)$
- 5) $\ln[64(z-4)^5]$
- 6) $\log\left(\frac{x^6}{36y^2}\right)$
- 7) $\frac{1}{5}$
- 8) 17
- 9) -.8
- 10) $x = \frac{ln16}{ln3}$
- 11) x = -3, x = -2
- 12) x = 0
- 13) $x = \frac{ln\frac{7}{6}}{ln2} - 5$
- 14) $x = e^{\frac{2}{3}}$
- 15) x = 2
- 16) x = 1
- 17) x = 10
- 18) ∅
- 19) $\mathrm{x} = \frac{1}{999}$
- 20) $x = \frac{7}{5}$
- 21) x = 1.585
- 22) x = 2
- 23) 480,732.12
- 24) r = 11%
- 25) (-1,-1)
- 26) No solution

Example Answers (cont.)

- 27) Infinitely many solutions
- 28) (-1,-1)
- 29) No solution
- 30) Infinitely many solutions
- 31) Amy's Income = $49,000, Vince's Income = $33,000
- 32) $6.50 per adult, $4 per child
- 33) (-1,-1)
- 34) No solution
- 35) Infinitely many solutions
- 36) $\begin{bmatrix} 5 & 2 \\ 0 & 8 \end{bmatrix}$
- 37) $\begin{bmatrix} -4 & 12 \\ 11 & 11 \\ -3 & -20 \end{bmatrix}$
- 38) $\begin{bmatrix} -2 & 14 \\ 18 & 12 \end{bmatrix}$
- 39) $\begin{bmatrix} -2 & 3 & 1 \\ -2 & 4 & 2 \\ -1 & 6 & 5 \end{bmatrix}$
- 40) Undefined.
  $2 \times 1\ matrix\ \cdot 2 \times 2\ matrix$
  doesn't work