Elementary Row Operations for Matrices

A. Introduction

A matrix is a rectangular array of numbers - in other words, numbers grouped into rows and columns. We use matrices to represent and solve systems of linear equations. For example, the system of equations

$$\begin{array}{rcrcrcl}
 & & 8y & + & 16z & = & 0 \\
x & & & - & 3z & = & 1 \\
-4x & + & 14y & + & 2z & = & 6
\end{array}
\longrightarrow$$

*Make sure to line up all variables and leave space if one is missing.

can be represented by what is called an augmented matrix as seen below:

$$\begin{array}{rc}
\begin{array}{r}
\text{Row 1 } (R_1) \rightarrow \\
\text{Row 2 } (R_2) \rightarrow \\
\text{Row 3 } (R_3) \rightarrow
\end{array} &
\left[\begin{array}{ccc|c}
0 & 8 & 16 & 0 \\
1 & 0 & -3 & 1 \\
-4 & 14 & 2 & 6
\end{array}\right] \\
 & \begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
x & y & z & \text{constant}
\end{array}
\end{array}$$

* Place a 0 in the matrix if the coefficient of a variable is 0.

Coefficients of the three unknown variables (x, y, and z) and the constant terms are placed in their respective places in the matrix.

Solving a system of equations using a matrix means using row operations to get the matrix into the form called *reduced row echelon form* like the example below:

$$\left[\begin{array}{ccc|c}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 6 \\
0 & 0 & 1 & 2
\end{array}\right]$$

* Make sure only ones are on the diagonal with 0's every other position except for the last column.

This column can have any numbers.

B. Row Operations

We can perform elementary row operations on a matrix to solve the system of linear equations it represents. There are three types of row operations.

1) **Interchanging two rows**

Rows can be moved around by switching any two. In this case, R_1 and R_2 have been switched.

$$\left[\begin{array}{ccc|c}
0 & 8 & 16 & 0 \\
1 & 0 & -3 & 1 \\
-4 & 14 & 2 & 6
\end{array}\right]
\xrightarrow{R_1 \leftrightarrow R_2}
\left[\begin{array}{ccc|c}
1 & 0 & -3 & 1 \\
0 & 8 & 16 & 0 \\
-4 & 14 & 2 & 6
\end{array}\right]$$

2) **Multiplying a row by a nonzero constant**

We can multiply any row by any number except 0. When a row is multiplied by a number, every element in that row must be multiplied by the same number. Below, R_2 is multiplied by 2.

$$\left[\begin{array}{ccc|c}
1 & 0 & -3 & 1 \\
0 & 8 & 16 & 0 \\
-4 & 14 & 2 & 6
\end{array}\right]
\xrightarrow{2\,R_2 \rightarrow R_2}
\left[\begin{array}{ccc|c}
1 & 0 & -3 & 1 \\
0 & 16 & 32 & 0 \\
-4 & 14 & 2 & 6
\end{array}\right]$$

3) **Adding a multiple of a row to another row**

We may also multiple a row by any number except 0 and add the results to another row.

Here, we multiplied R_1 by 4 and added the answer to R_3 to get a new row R_3.

$4R_1$	=	4	0	-12	4
$+R_3$	=	-4	14	2	6
R_3		0	14	-10	10

Our new matrix looks like this:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 8 & 16 & 0 \\ -4 & 14 & 2 & 6 \end{array}\right] \xrightarrow{4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 8 & 16 & 0 \\ 0 & 14 & -10 & 10 \end{array}\right]$$

Note: R_1 never changed. We only multiplied R_1 by 4 so that we could add the product to R_3. There was no intention of keeping the product as part of the new matrix.

C. Solving a System of Equations using a Matrix

We will use the original matrix presented in part A. Our goal is to get the matrix in the *reduced row echelon form* that we discussed previously. The first step in solving our matrix is to "work out" the first column. This means that the column must contain only one 1 and the rest 0's before we can continue on to the next column.

1) Finding the 1

The first step is to get the 1 in the desired location. In this case, the upper left corner. To make it easier on ourselves we first look to see if we can interchange rows so we will have a 1 in the first position. Since R_2 already has a 1 in the first position, we switch it with R_1 using the interchanging row operation.

$$\left[\begin{array}{ccc|c} 0 & 8 & 16 & 0 \\ 1 & 0 & -3 & 1 \\ -4 & 14 & 2 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 8 & 16 & 0 \\ -4 & 14 & 2 & 6 \end{array}\right]$$

$\uparrow$

Always start with this column.

2) Filling with 0's

We must fill the rest of the column with 0's. Since R_2 already has a 0, nothing needs to be done. However, the –4 in R_3 must be changed. (The work for this step was done in part 3 of section B.) We multiply R_1 by 4 and add the results to R_3, giving us a new R_3.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 8 & 16 & 0 \\ -4 & 14 & 2 & 6 \end{array}\right] \xrightarrow{4R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 8 & 16 & 0 \\ 0 & 14 & -10 & 10 \end{array}\right]$$

Note: You always use the row with the 1 for the multiplication, and you multiply by the opposite of the number that you are changing to 0.

3) Finish the Matrix

Now that the first column is finished, we can work on the next column applying the same procedures.

We need a 1 in the second row second position. The easiest way to accomplish this is to use the second type of row operation. Multiply the second row by $\frac{1}{8}$ because $\frac{1}{8} \cdot 8 = 1$.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 8 & 16 & 0 \\ 0 & 14 & -10 & 10 \end{array}\right] \xrightarrow{\frac{1}{8}R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 14 & -10 & 10 \end{array}\right]$$

↑ The next column to be worked out.

Continue repeating these steps as needed until the matrix is completed.

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 14 & -10 & 10 \end{array}\right] \rightarrow \underset{-14R_2 + R_3 \rightarrow R_3}{\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -38 & 10 \end{array}\right]} \rightarrow \underset{-\frac{1}{38}R_3 \rightarrow R_3}{\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{5}{19} \end{array}\right]}$$

$$\rightarrow \underset{-2R_3 + R_2 \rightarrow R_2}{\left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & \frac{10}{19} \\ 0 & 0 & 1 & -\frac{5}{19} \end{array}\right]} \rightarrow \underset{3R_3 + R_1 \rightarrow R_1}{\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{19} \\ 0 & 1 & 0 & \frac{10}{19} \\ 0 & 0 & 1 & -\frac{5}{19} \end{array}\right]}$$

Recall that the first column represents the x, the second y, the third z. We can rewrite the matrix back to a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{19} \\ 0 & 1 & 0 & \frac{10}{19} \\ 0 & 0 & 1 & -\frac{5}{19} \end{array}\right] \rightarrow \begin{array}{l} 1x + 0y + 0z = \frac{4}{19} \\ 0x + 1y + 0z = \frac{10}{19} \\ 0x + 0y + 1z = -\frac{5}{19} \end{array}$$

So, the solution to this system of the linear equations is $x = \frac{4}{19}$, $y = \frac{10}{19}$, and $z = -\frac{5}{19}$