Related Rates

GUIDELINES FOR SOLVING A RELATED-RATES PROBLEM

1. Identify all known and unknown quantities and variables. A particular rate should be given. Identify this rate along with the rate you are calculating. (The two variables that are in these two rates must have an equation that relates them and ONLY them. No other variables can be in the equation for Step 4.)
2. Make a sketch, if one is not given, and label it appropriately.
3. Write an equation involving the variables from Step 1 whose rates of change are known and unknown. (The two variables that are in these two rates must have an equation that relates them and ONLY them. No other variables can be in the equation for Step 4. If the equation contains more that these two variables, there must be a substitution that can be made to eliminate a variable.) See examples for more clarification.
4. Implicitly differentiate both sides of the equation with **respect to time**.
5. Substitute known values for the variables and the rates of change given in the problem and solve for the unknown rate.

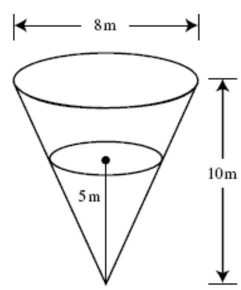
Example 1:

> Water pours into a conical tank whose height is 10 meters and whose base has a radius of 4 meters at a rate of 6 m³/min. At what rate is the water level rising when the water level is 5m high?

Step 1. Identify all known and unknown quantities and variables:

$$\frac{dV}{dt} = 6 \quad height\ of\ water = 5 \quad radius\ of\ cone = 4 \quad height\ of\ cone = 10 \quad \frac{dh}{dt} = ?$$

Step 2. Make a sketch and label it appropriately.

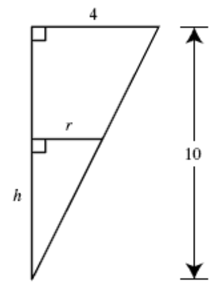


Step 3. Write an equation involving the variables from Step 1's rates.

We are looking for $\frac{dh}{dt}$ and $\frac{dV}{dt}$ is given, so we must find an equation relating V and h:

$$\mathrm{V} = \frac{1}{3}\pi r^2 h \quad \text{(volume of a cone)}$$

But this equation also contains another variable, r. We need a relationship from what was given between h and r. Using a cross-section of half of the cone we can see similar triangles:



We can use these similar triangles to establish a relationship between r and h:

$$\frac{r}{h} = \frac{4}{10} \Rightarrow r = 0.4h$$

Substituting using the relationship between r and h, the equation would now be in terms of V and h only:

$$\mathrm{V} = \frac{1}{3}\pi(0.4h)^2 h \Rightarrow \mathrm{V} = \frac{.16}{3}\pi h^3$$

Step 4. Implicitly differentiate with respect to time.

$$\frac{dV}{dt} = 0.16\pi h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{0.16\pi h^2}$$

Step 5. Substitute given values and then solve.

In this problem $\frac{dV}{dt} = 6$ and the height of the water , h = 5.

$$\frac{dh}{dt} = \frac{6}{0.16\pi(5)^2} \Rightarrow \frac{dh}{dt} \approx .48\frac{m}{min}$$

Example 2:

A 13ft ladder is leaning against a wall when its base starts to slide away. By the time the base is 12ft from the wall, the base is moving at the rate of 5ft/sec. How fast is the top of the ladder sliding down the wall then? At what rate is the area of the triangle formed by the ladder, wall, and ground changing? At what rate is the angle α between the ladder and the ground changing then?

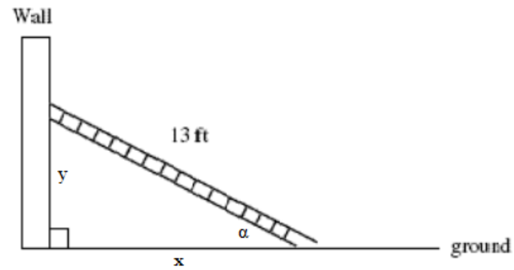
(a) How fast is the top of the ladder sliding down the wall?

Step 1. Identify all known and unknown quantities and variables.

$$base = x = 12 \qquad \frac{dx}{dt} = 5 \qquad ladder = r = 13$$

$$height\ top\ of\ ladder\ is\ up\ the\ wall = y = ? \qquad \frac{dy}{dt} = ?$$

Step 2. Make a sketch and label it appropriately.



Step 3. Write an equation involving the variables from Step 1's rates.

We are looking for $\frac{dy}{dt}$ and $\frac{dx}{dt}$ is given so we must find an equation relating x and y:

$$x^2 + y^2 = r^2 \quad \text{(Pythagorean Theorem)} \Rightarrow \quad x^2 + y^2 = 13^2$$

Step 4. Implicitly differentiate with respect to time.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y}$$

Step 5. Substitute given values and then solve.

In this problem, $\frac{dx}{dt} = 5$ and since we have a right triangle, we also know the value of y when x = 12,

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad 12^2 + y^2 = 13^2 \quad \Rightarrow \quad y = 5$$

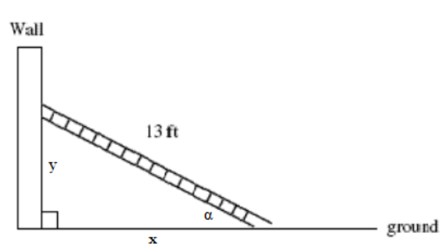
$$\frac{dy}{dt} = \frac{-(12)(5)}{5} \quad \Rightarrow \quad \frac{dy}{dt} = -12\frac{ft}{sec}$$

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing?

Step 1. Identify all known and unknown quantities and variables.

$\frac{dA}{dt} = ?$ $\qquad x = 12$ $\qquad \frac{dx}{dt} = 5$ $\qquad r = 13$ $\qquad y = 5$ $\qquad \frac{dy}{dt} = -12$

Step 2. Make a sketch and label it appropriately.



Step 3. Write an equation involving the variables from Step 1's rates.

We are looking for $\frac{dA}{dt}$. $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are given so we must find an equation relating A, x and y:

$$A = \frac{1}{2}xy$$

Step 4. Implicitly differentiate with respect to time.

$$\frac{dA}{dt} = \frac{1}{2}[x\frac{dy}{dt} + y\frac{dx}{dt}]$$

Step 5. Substitute given values and then solve.

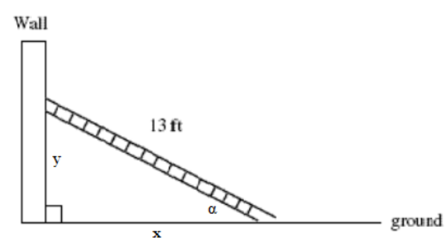
$$\frac{dA}{dt} = \frac{1}{2}[(12)(-12) + (5)(5)] \Rightarrow \frac{dA}{dt} = \frac{-119}{2}\frac{ft^2}{sec} = -59.5\ \frac{ft^2}{sec}$$

(c) At what rate is the angle α between the ladder and the ground changing then?

Step 1. Identify all known and unknown quantities and variables.

$x = 12$ $\frac{dx}{dt} = 5$ $\frac{d\alpha}{dt} = ?$ $r = 13$ $y = 5$ $\frac{dy}{dt} = -12$

Step 2. Make a sketch and label it appropriately.



Step 3. Write an equation involving the variables from Step 1's rates.

Any trigonometric function involving α would work. We are looking for $\frac{d\alpha}{dt}$.

$$\sin\alpha = \frac{y}{r} \Rightarrow \sin\alpha = \frac{y}{13}$$

Step 4. Implicitly differentiate with respect to time.

$$cos\,\alpha\frac{d\alpha}{dt} = \frac{1}{13}\frac{dy}{dt} \Rightarrow \frac{d\alpha}{dt} = \frac{(\frac{1}{13})\frac{dy}{dt}}{cos\,\alpha}$$

Step 5. Substitute given values and then solve. $cos\,\alpha = \frac{adj}{hyp} = \frac{x}{r} = \frac{12}{13}$ and $\frac{dy}{dt} = -12$ was given:

$$\frac{d\alpha}{dt} = \frac{(\frac{1}{13})(-12)}{\frac{12}{13}} \Rightarrow \frac{d\alpha}{dt} = -1\ \frac{rad}{sec}$$